Exam 1

October 3, 2008

Please budget your time in solving these problems. Make sure you attempt each problem in order to maximize partial credit. *Please show your work in your blue book.* Do not spend excessive time on any single part of a problem. There are 3 problems and 100 points possible.

You may find the following relations useful:

$$\int_0^\infty e^{-x^2/b^2} dx = \frac{b}{2}\sqrt{\pi} \tag{1}$$

$$\int_0^\infty \frac{(1-e^{-x^2})^2}{x^2} dx = 2\sqrt{\pi}(1-\frac{1}{\sqrt{2}}) \tag{2}$$

1. (24 points) In spherical coordinates define a vector field $\vec{V} \equiv \hat{\phi}$ where $\hat{\phi}$ is the unit vector along the azimuthal coordinate.

   (a) (7 pts) Evaluate the path integral:

   $$\int_P \vec{V} \cdot d\vec{\ell}$$

   for a path P defined as a counterclockwise circle of radius R in the x-y plane and centered on the origin.

   (b) (10 pts) Evaluate the curl of $\vec{V}$: $\nabla \times \vec{V}$. (You may leave your answer in purely spherical coordinates and unit vectors, although the result can be expressed more compactly as a mixture of spherical and cartesian factors.)

   (c) (7 pts) Use the results of parts (a) and (b) to verify explicitly Stokes's theorem for a planar disk bounded by the path P.

2. (48 points) Imagine that all space is pervaded by an electric field of the form $\vec{E}(\vec{r}) = \frac{Q_0}{4\pi\epsilon_0}\left(\frac{1-e^{-r^2/a^2}}{r^2}\right)\hat{r}$.

   (a) (10 pts) Find the corresponding electric charge density $\rho(\vec{r})$.

   (b) (12 pts) Integrate $\rho(\vec{r})$ to obtain the total charge in a spherical volume of radius R centered on the origin. What is your result for $R \to \infty$?

   (c) (14 pts) Find the total electrostatic energy W required to assemble this charge distribution by bringing in one infinitesimal spherical layer after another, as in problem 2.33.

   (d) (12 pts) Check your answer to part (c) by integrating the electric field's energy density over all space.

3. (28 points) A point charge q_0 lies **inside** a grounded spherical conducting surface of radius R centered on the origin. The charge is placed on the z axis at $z = \lambda R$, where λ is a dimensionless number satisfying $0 \leq \lambda \leq 1$. The resulting potential inside the sphere can be found via the method of images and expressed (try showing as an exercise <u>after</u> the exam) as

$$V(r,\theta) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + \lambda^2 R^2 - 2\lambda R r\cos(\theta)}} + \frac{-1}{\sqrt{\lambda^2 r^2 + R^2 - 2\lambda R r\cos(\theta)}}\right]$$

where the first term can be attributed to q_0 and the second term to its image charge.

(a) (6 pts) Evaluate the expression for $V(r,\theta)$ in the limit $\lambda \to 0$ and explain physically why the result makes sense.

(b) (6 pts) Evaluate the expression for $V(r,\theta)$ in the limit $\lambda \to 1$ and explain physically why the result makes sense.

(c) (12 pts) Find the surface charge density $\sigma(\theta;\lambda)$ on the inside surface of the sphere.

(d) (4 pts) Evaluate your result from part (c) in the limit $\lambda \to 0$ and explain physically why the result makes sense.