Physics 401 – Fall 2007

Practice Final Exam

Please budget your time in solving these problems. Make sure you attempt each problem in order to maximize partial credit. *Please show your work in your blue book.* Do not spend excessive time on any single part of a problem. There are 5 problems and 150 points possible. Please pay attention to the points assigned to each problem in allocating your time.

You may find the following relations useful:

$$\begin{aligned}
\int e^{-x/a} dx &= -a\, e^{-x/a} \\
\int x\, e^{-x/a} dx &= -(ax + a^2)\, e^{-x/a} \\
\int x^2\, e^{-x/a} dx &= -(ax^2 + 2\,a^2 x + 2\,a^3)\, e^{-x/a} \\
\int_0^\infty x^n\, e^{-x/a} dx &= n!\, a^{n+1}
\end{aligned}$$

1. (20 points) A critically damped harmonic oscillator is described by the following equation of motion:

$$\ddot{x} + 2\,\beta\,\dot{x} + \omega_0^2\,x = 0.$$

The oscillator is kicked sharply so that at time $t = 0$ its displacement from equilibrium remains $x(0) = 0$, but its velocity is $\dot{x}(0) = v_0$.

   (a) (10 pts) Find $x(t)$ for $t > 0$.

   (b) (10 pts) Find the maximum displacement x_{max} for $t > 0$. Sketch $x(t)$ *vs.* t, indicating the time of maximum displacement.

2. (20 points) A particle is projected vertically upward to a height h above the Earth's surface at a northern latitude λ. Find its deflection along the East-West direction (East $\equiv$ positive) *at the top of its trajectory.* Neglect air resistance and assume only small vertical heights.

3. (30 points) Consider a symmetric, homogenous plate with principal moments of inertia $I_1 = I_2 \equiv I$ and $I_3 = 2\,I$. At time $t = 0$, its angular velocity $\vec{\omega}(0)$ has components along the principal axes given by $(\omega_{1_0},\, \omega_{2_0},\, \omega_{3_0})$. The plate rotates in a viscous gas such that the resistive torque acting on the plate is $\vec{N}(t) = -\lambda I \vec{\omega}(t)$ where the damping parameter λ is real and positive.

   (a) (14 pts) For this general case, find $\omega_3(t)$ in the body-frame of the plate. It is *not* necessary to find $\omega_1(t)$ and $\omega_2(t)$.

   (b) (14 pts) Now consider the special case $\omega_{3_0} = 0$, but $\omega_{1_0} \neq 0$ and $\omega_{2_0} \neq 0$. Find $\omega_1(t)$ and $\omega_2(t)$. Do $\omega_1(t)$ and $\omega_2(t)$ have oscillatory terms? Why or why not?

   (c) (2 pts) Comment on the *stability* of rotation in part (b).

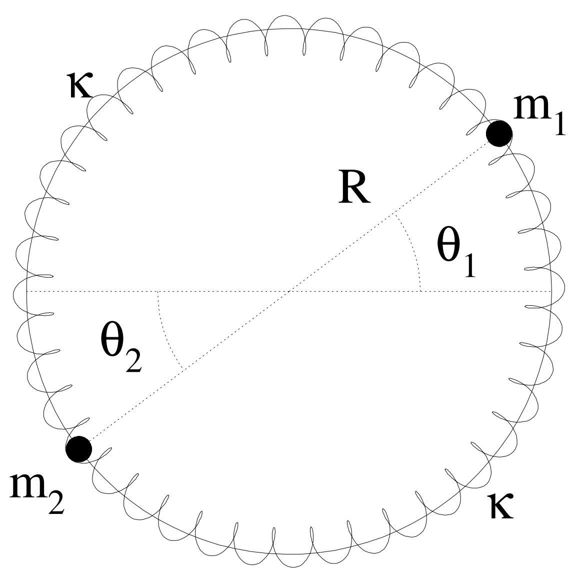
4. (45 points) Consider two beads of mass m_1 and m_2 initially placed opposite to each other on a horizontal circular ring of radius R, as shown in the figure below. Two identical springs of force constant κ connect the masses, where the springs wrap around the wire in opposite arcs, each of relaxed length $\ell = \pi R$. Denote the positions of the beads around the ring by angles θ_1, and θ_2, which are both defined to be zero when the masses are in equilibrium (180 degrees apart) and to be positive in the counter-clockwise direction.

   (a) (5 pts) Write down the Lagrangian for this system in terms of the two coordinates θ_1, θ_2 and their time derivatives.

   (b) (10 pts) From this Lagrangian, find the equations of motion for θ_1 and θ_2.

   (c) (18 pts) Assume oscillatory solutions and determine the two eigenfrequencies of the system.

   (d) (12 pts) Construct orthogonal eigenvectors for these eigenfrequencies. Describe the normal modes of the system. Explain why the higher eigenfrequency value makes sense, given the geometry.



5. (35 points) In the following assume a spherically symmetric, infinite mass distribution centered on the origin with a mass density function

$$\rho(r) = \rho_0 \, e^{-r/a}$$

where ρ_0 is the density at the origin and a is a positive, real distance parameter.

(a) (10 pts) Find the total mass M_0 of this distribution.

(b) (25 pts) Find its gravitational self-energy U. Express your result in terms of M_0, *i.e.*, eliminate ρ_0. *Tip: To reduce the chance of algebraic error, it pays to check the dimensions of each term in your expressions. For example, it rarely makes sense to add a distance cubed (volume) to a distance squared (area).*