Physics 401 – Fall 2007

Exam 2

Please budget your time in solving these problems. Make sure you attempt each problem in order to maximize partial credit. *Please show your work in your blue book.* Do not spend excessive time on any single part of a problem. There are 3 problems and 100 points possible.

Notes:

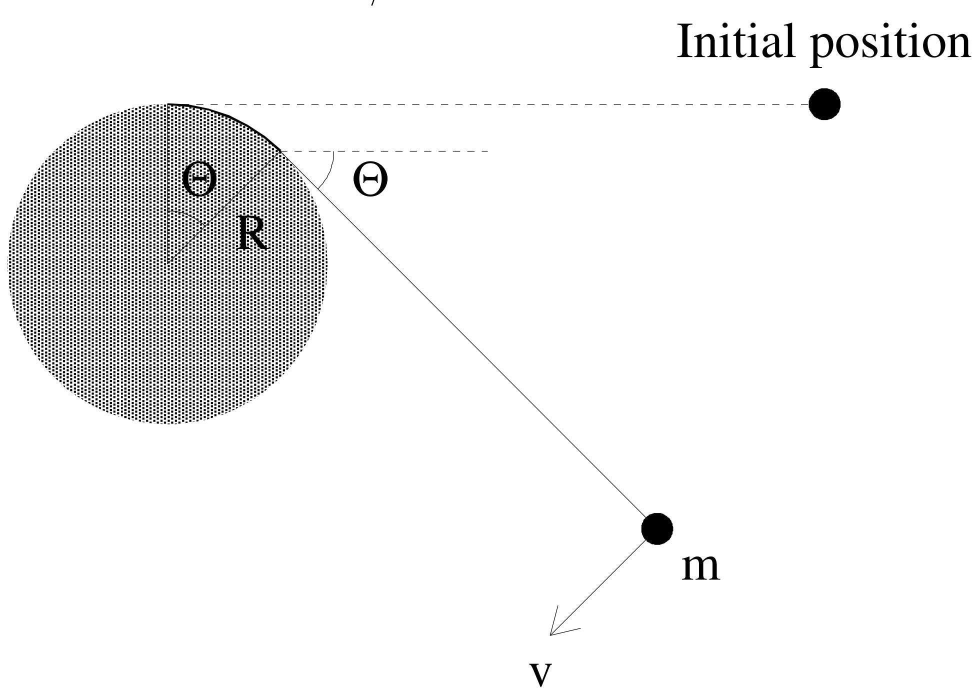
- If a later part of a problem uses an expression from an earlier part that you are unable to obtain, explain briefly how you *would* solve the later part if you had the missing expression.
- Similarly, if you are asked to compare a result to expectation in a certain limit and your result disagrees with expectation, be sure to explain (briefly) what that expectation is.

1. (30 points) A pendulum is constructed by attaching a mass m to an extensionless string of length ℓ. The upper end of the string is connected to the uppermost point on a vertical disk of radius R ($\frac{3}{2}\pi R < \ell < 2\pi R$ – see figure). The mass is held taut horizontally from the top of the cylinder ($\theta = 0$) and released.

   (a) (10 pts) Use energy conservation to obtain the speed v of the mass as a function of θ after release.

   (b) (12 pts) Use Newtonian concepts to find the tension in the string as a function of θ.

   (c) (8 pts) Find the maximum string length ℓ for which the string remains taut for $\theta = 3\pi/2$.



2. (35 points) Consider a damped harmonic oscillator with the equation of motion:

$$m\ddot{x} \quad = -kx - b\dot{x}$$

(a) (15 pts) Show that the effective Lagrangian:

$$L \quad = \quad e^{\lambda t}\,(\frac{1}{2}m(\dot{x})^2 - \frac{1}{2}k(x)^2)$$

can be used to derive the correct equation of motion for x and determine the necessary value for the constant λ.

(b) (8 pts) Construct the corresponding Hamiltonian $H(x, p)$ where p is the generalized momentum for x (not the physical momentum).

(c) (12 pts) Derive Hamilton's equations from the Hamiltonian. Combine these two 1st-order differential equations to obtain a single 2nd order differential equation for x.

3. (35 points) A useful model of gravitational interactions is provided by the "Vortex" toy in which a two-dimensional surface is shaped into a funnel so that a coin rolling on the surface in a uniform gravitational field travels on a trajectory analogous to that of planetary orbits. Because of friction, the coin gradually loses energy and spirals toward the center as its average height decreases (see figure).

   Let's analyze this model with some simplifications to reduce algebra. Treat the surface as frictionless and assume a sliding (not rolling) point mass m for which rotational kinetic energy about its center of mass can be neglected. Assume the surface height is described in cylindrical coordinates by

$$z(x,y) = -\frac{K}{\sqrt{x^2+y^2}} = -\frac{K}{r}$$

   (a) (8 pts) Write down the Lagrangian for the mass m in cylindrical coordinates, taking into account the constraint that the mass remains in contact with the surface. (Absorb the constraint implicitly into the Lagrangian to obtain a minimum set of generalized coordinates; do <u>not</u> use an explicit Lagrange multiplier constraint.)

   (b) (7 pts) Derive the Lagrange equation for the cylindrical azimuthal coordinate ϕ. What is the physical interpretation of this equation?

   (c) (20 pts) Derive the Lagrange equation for the cylindrical radial coordinate r. Eliminate any ϕ, $\dot{\phi}$ dependence, using the equation of motion from part (b). Is this model mathematically equivalent to a true gravitational central force?